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Dissipative heating in convective flows

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Dissipative heating is produced by irreversible processes, such as viscous or ohmic heating, in a convecting fluid; its importance depends on the ratio d/H_T of the depth of the convecting region to the temperature scale height. Integrating the entropy equation for steady flow yields an upper bound to the total rate of dissipative heating in a convecting layer. For liquids there is a regime in which the ratio of dissipative heating to the convected heat flux is approximately equal to $c(d/H_T)$, where the constant c is independent of the Rayleigh number. This result is confirmed by numerical experiments using the Boussinesq approximation, which is valid only if d/H_T is small. For deep layers the dissipative heating rate may be much greater than the convected heat flux. If the earth's magnetic field is maintained by a convectively driven dynamo, ohmic losses are limited to 5 % of the convected flux emerging from the core. In the earth's mantle viscous heating may be important locally beneath ridges and behind island arcs.

1. Introduction

Some source of energy is needed to maintain the motion of a real fluid. In a convecting region energy is supplied externally (by heating from below) or internally (by radioactive heating) and carried to the outer boundary. In a steady state the overall rate of working by the pressure forces must equal the rate at which energy is dissipated by irreversible processes. Heat generated by these processes contributes to the local energy balance but does not affect the net flux of energy that emerges. For many fluid-dynamical problems, where the Boussinesq approximation is valid, dissipative heating is negligible compared with the heat flux through the region. However, this is not true for astrophysical convection and, as Tozer (1965) recognized, may not hold for convection in the earth's mantle.

The aim of this paper is to examine the energetics of convection in a compressible fluid. We shall in particular refer to models of convection in the earth's mantle (McKenzie, Roberts & Weiss 1974, hereafter referred to as I). In addition, we shall try to establish a correct upper bound to the rate of ohmic heating in the core (assuming that convection drives the dynamo which maintains the earth's magnetic field).

To clarify the problem, consider for the moment the simplest case, of steady convection in a layer of viscous fluid heated uniformly from below. Let T_u and T_l be the absolute temperatures of the upper and lower boundaries ($T_l > T_u$). Then conservation of energy (the first law of thermodynamics) demands that the heat fluxes across the upper and lower boundaries must be equal. If work is somehow extracted from the system we might regard it as a heat engine. Let W be the rate of working and $\mathscr{F}_u$ and $\mathscr{F}_l$ the total heat fluxes across the upper and lower boundaries; then $W = \mathscr{F}_l - \mathscr{F}_u$ and the second law of thermodynamics tells us that $W/\mathscr{F}_l < (T_l - T_u)/T_l$. None of these statements involves the rate of viscous dissipation Φ for viscous heating contributes only to the internal energy budget. What then can be said about the ratio of Φ to $\mathscr{F}_l$?

More generally, Φ is the rate of heating by irreversible processes (including both shear-stress and ohmic heating). In § 2 we use the entropy equation to set an upper bound to dissipative heating when $W = 0$ and show that

$$\Phi/\mathscr{F}_l < (T_l - T_u)/T_u. \tag{1}$$

Thus the dissipative heating rate may exceed the heat flux through the layer if $T_l \gg T_u$. A necessary, but not sufficient, condition for the Boussinesq approximation to hold is that the layer depth d should be small compared with the temperature scale height H_T [defined in equation (23) below]. Under the conditions usually assumed in deriving the Boussinesq approximation it follows that

$$\Phi/\mathscr{F}_l = O(d/H_T) \ll 1 \quad \text{(Malkus 1964).}$$

In the next section we obtain a more stringent estimate of the shear-stress heating in a liquid with a small coefficient of expansion α, such that $\alpha T_l \ll 1$. Under certain conditions the ratio

$$\Phi/\mathscr{F}_l \doteqdot d/H_T \tag{2}$$

and is independent of the Rayleigh number. This result confirms that the global effect of shear-stress heating is small when $d \ll H_T$ but indicates that it may be large if the layer is deep compared with the temperature scale height. Malkus (1973) considered the rate of dissipation in a Boussinesq fluid in the context of a more complicated problem. His approach confirms that the ratio of the global dissipation rate to the average convected flux is equal to d/H_T in the Boussinesq limit. The numerical experiments described in I employed the Boussinesq approximation and shear-stress heating was ignored; for the parameters used there to describe the earth's mantle the ratio $d/H_T \approx 0{\cdot}12$. In § 4 we compute the local rate of shear-stress heating for various models. This heating is concentrated into narrow regions, corresponding to rising and sinking plumes, and the concentration is particularly marked for convection driven by heating from within. The global heating rates computed for these models are proportional to the heat flux and independent of the Rayleigh number, as predicted by (2). The difficulty of extending this treatment to gases is briefly discussed in § 5.

Hydromagnetic dynamos, driven by convection, have been invoked to maintain planetary and stellar magnetic fields. Here viscous dissipation is small compared with ohmic heating. In § 6 we derive expressions analogous to (1) and (2) for spherical dynamos (Malkus 1973). The rate of ohmic dissipation may exceed the thermal flux if the temperature scale height is small; since the heat produced remains in the system, the dynamo cannot be regarded as a heat engine whose thermodynamic efficiency might limit Φ. It is only in the Boussinesq limit, when $T_l \approx T_u$, that (1) resembles the upper bound on W.

Finally, in §7, we consider some geophysical implications of these results. Shear-stress heating may provide local increases in the temperature of the upper mantle, which could explain the high heat flow behind island arcs as well as the variable composition of oceanic basalts erupted at ridges. Previous estimates of ohmic dissipation in the core (though based on faulty arguments) are not significantly affected.

Relatively little work has yet been carried out on shear-stress heating in convective flows. Gebhart (1962) and Gebhart & Mollendorf (1969) employed boundary-layer expansions to show that this heating became important at high Reynolds numbers if d was comparable with H_T. However, Ackroyd (1974) has pointed out that they used the Boussinesq form of the boundary-layer equations, valid only if $H_T \gg d$, and that their results are considerably modified when the full equations are used. Apart from an erroneous paper by Rice (1971) the only treatment of viscous dissipation in Bénard convection is due to Turcotte *et al.* (1974). They adopted modified Boussinesq equations in which both viscous heating and the adiabatic gradient appeared in the heat transport equation, though the flow remained incompressible. Solutions were computed, at infinite Prandtl number, for $d/H_T \leqslant 3$; however, the ratio $\Phi/\mathscr{F}_l$ was not calculated. A proper investigation of convection in a liquid when $d/H_T \gtrsim 1$ requires a self-consistent anelastic model in which all the relevant non-Boussinesq effects are retained: in particular, the velocity is no longer solenoidal.

2. Integral constraints

In this section we obtain an upper bound for the dissipative heating rate in a steadily convecting fluid that is valid for any equation of state or stress–strain relationship. In a Boussinesq fluid, with $d \ll H_T$, this heating rate is small. We obtain two integral constraints by considering the energy equation only, without explicitly invoking the equation of motion.

Consider a convecting fluid occupying a volume V enclosed by a surface S on which the normal component of the velocity $\mathbf{u}$ is zero and either $\mathbf{u}$ itself or the tangential stress vanishes. We suppose that the fluid is conducting, with a magnetic field $\mathbf{B}$ maintained by electric currents flowing in V. Locally, conservation of energy requires that the rate of change of the total energy (internal, kinetic, electromagnetic and potential) is equal to the net inward flux of energy plus the rate of internal generation of heat by chemical and nuclear reactions. In the magnetohydrodynamic approximation

$$\frac{\partial}{\partial t}\left(\rho e + \tfrac{1}{2}\rho u^2 + \frac{B^2}{2\mu_0} - \rho\Psi\right) = -\frac{\partial}{\partial x_i}\left[\rho(e + \tfrac{1}{2}u^2 - \Psi)u_i + \frac{\epsilon_{ijk}E_jB_k}{\mu_0} + Pu_i - \tau_{ij}u_j - k\frac{\partial T}{\partial x_i}\right] + H \qquad (3)$$

(Landau & Lifshitz 1959, 1960). Here ρ is the density, e the internal energy, T the absolute temperature, k the thermal conductivity, H the volumetric heating rate and Ψ the gravitational potential, such that the gravitational acceleration $\mathbf{g} = \nabla\Psi$; P is the thermodynamic pressure and $\tau_{ij} - P\delta_{ij}$ the total stress tensor, so that τ_{ij} is the contribution from irreversible processes (Batchelor 1967). The right-hand side of (3) includes the Poynting flux $\mathbf{E} \wedge \mathbf{B}/\mu_0$, where $\mathbf{E}$ is the electric field and μ_0 the permeability (assumed constant).

The primary global constraint is the conservation of energy. In a steady state we integrate (3) over V to obtain

$$\int_S k\frac{\partial T}{\partial x_i}\,dS_i + \int_V H\,dV = 0. \tag{4}$$

(Here we have assumed that the electric current $\mathbf{j}$ vanishes everywhere outside V, so that

$$\int_S \mathbf{E} \wedge \mathbf{B}\,.\,d\mathbf{S} = 0.)$$

The net heat flux out of the region is equal to the total rate of internal heating. In particular, for steady convection in a horizontal layer, the difference between the fluxes across the upper and lower boundaries equals the rate at which heat is generated in the layer. It must be emphasized that dissipative heating does not appear in (4); irreversible processes such as shear-stress or ohmic heating do not contribute to the overall heat flux.

The most general constraint involving dissipative heating may be obtained from (1) by introducing the entropy s and substituting for $\mathbf{E}$ from Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \wedge \mathbf{B}), \tag{5}$$

where σ is the electrical conductivity. Then

$$\rho T\left(\frac{\partial s}{\partial t} + \mathbf{u}\,.\,\nabla s\right) = \nabla\,.\,(k\nabla T) + H + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{j^2}{\sigma} \tag{6}$$

(Landau & Lifshitz 1960). Irreversible processes, such as shear-stress or ohmic heating, provide positive source terms, similar to H, for the entropy. For a steady flow conservation of mass implies that

$$\nabla\,.\,(\rho\mathbf{u}) = 0. \tag{7}$$

Hence (6) may be divided by T and integrated over the volume V to yield

$$\int_V \frac{1}{T}\nabla\,.\,(k\nabla T)\,dV + \int_V \frac{1}{T}\left(H + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{j^2}{\sigma}\right)dV = \int_S s\rho\mathbf{u}\,.\,d\mathbf{S} = 0. \tag{8}$$

Rewriting (8), we have

$$\int_S \frac{k}{T}\nabla T\,.\,d\mathbf{S} + \int_V k\left|\frac{\nabla T}{T}\right|^2 dV + \int_V \frac{1}{T}\left(H + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{j^2}{\sigma}\right)dV = 0. \tag{9}$$

We now consider Rayleigh–Bénard convection in a plane layer. Let $\mathscr{F}_u$, T_u, $\mathscr{F}_l$ and T_l be the fluxes and absolute temperatures at the upper and lower boundaries. Since (6) allows both local cooling (e.g. by adiabatic expansion) and local heating, the temperature is not constrained to lie between T_u and T_l. Let T_m and T_c be the maximum and minimum temperatures in V, so that

$$T_c \leqslant T \leqslant T_m, \quad T_u \geqslant T_c, \quad T_l \leqslant T_m. \tag{10}$$

Now from (4)

$$\mathscr{F}_u = \mathscr{F}_l + Q, \tag{11}$$

where the global rate of internal heat generation

$$Q = \int_V H\,dV. \tag{12}$$

The surface integral in (9) becomes

$$\frac{\mathscr{F}_l}{T_l} - \frac{\mathscr{F}_u}{T_u} \geqslant \frac{\mathscr{F}_l}{T_m} - \frac{\mathscr{F}_u}{T_u}. \tag{13}$$

Since the second integral in (9) is positive definite we can derive the inequality

$$\int_V \frac{1}{T}\left(\tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{j^2}{\sigma}\right) dV < \frac{\mathscr{F}_u}{T_u} - \frac{\mathscr{F}_l}{T_l} - \int_V \frac{H}{T}\,dV \leqslant \mathscr{F}_u\left(\frac{1}{T_u} - \frac{1}{T_m}\right), \tag{14}$$

from (11). Finally, since $T \leqslant T_m$, (14) implies that the total rate of dissipative heating

$$\Phi = \int_V \left(\tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{j^2}{\sigma}\right) dV < \frac{\Delta T}{T_u}\mathscr{F}_u, \tag{15}$$

where

$$\Delta T = T_m - T_u \geqslant T_l - T_u. \tag{16}$$

In a viscous fluid without a magnetic field, (15) allows the overall rate of shear-stress heating Φ to exceed the flux $\mathscr{F}_u$ if $T_l > 2T_u$. Of course this is only an upper bound. Nevertheless it is instructive to compare this result with the corresponding limit to the power W that could be extracted from a convectively driven heat engine in an inviscid fluid. The local rate of working appears in the equations as a negative heat source. From (11)

$$\mathscr{F}_u = \mathscr{F}_l - W > 0 \tag{17}$$

and so $W < \mathscr{F}_l$ whereas, from (15), if $Q = 0$, $\Phi < (\Delta T/T_u)\mathscr{F}_l$. To be sure, the rate of working must satisfy the more restrictive thermodynamic constraint on the efficiency of a heat engine, that

$$W/\mathscr{F}_l < \Delta T/T_l < 1. \tag{18}$$

It is clear that heat generated by shear stresses, which remains in the region V, is quite different from work done outside the region.

The entropy equation (6) can alternatively be written in terms of the temperature as

$$\rho C_p\left[\frac{\partial T}{\partial t} + u_i\left\{\frac{\partial T}{\partial x_i} - \left(\frac{\partial T}{\partial x_i}\right)_s\right\}\right] - \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) - H = \tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{j^2}{\sigma}, \tag{19}$$

where the adiabatic temperature gradient

$$\left(\frac{\partial T}{\partial x_i}\right)_s = \frac{\alpha T}{\rho C_p}\frac{\partial P}{\partial x_i}, \tag{20}$$

α is the coefficient of thermal expansion and C_p is the specific heat at constant pressure, assumed constant. Integrating (19) over V for a steady state now yields

$$\int_V \left(\tau_{ij}\frac{\partial u_i}{\partial x_j}+\frac{j^2}{\sigma}\right)dV+\int_V \alpha T u_i \frac{\partial P}{\partial x_i}dV = \int_V \rho C_p u_i \frac{\partial T}{\partial x_i}dV - \int_S k\frac{\partial T}{\partial x_i}dS_i - Q. \quad (21)$$

From (21), with (4), (12), (7) and the boundary conditions on $\mathbf{u}$,

$$\Phi + \int_V \alpha T\mathbf{u}\,.\,\nabla P\,dV = 0. \quad (22)$$

The global rate of dissipative heating is exactly cancelled by the work done against the adiabatic gradient.

For the Boussinesq approximation to be valid fluctuations in thermodynamic quantities must be small and the layer depth d must be small compared with the temperature scale height

$$H_T = C_p/g\alpha. \quad (23)$$

In an unpublished paper W. V. R. Malkus showed that the terms on the right-hand side of (19) are of order d/H_T and can therefore be neglected in the Boussinesq approximation (see Malkus 1964, 1973). It follows also that $T_m = T_l$ and that $\Delta T \ll T_l$ when the Boussinesq approximation holds. Hence $T_u \approx T_l$ and the inequality (15) becomes, approximately,

$$\Phi/\mathscr{F}_u < \Delta T/T_l \ll 1. \quad (24)$$

But this condition, which resembles (18), holds only when $d \ll H_T$.

3. Shear-stress heating in a liquid

Under certain conditions, which apply in the earth's mantle, (22) can be simplified to give a more useful, if less general, result. The pressure gradient can be separated into two parts:

$$\nabla P = \nabla P_0 + \nabla P_1, \quad (25)$$

where the hydrostatic gradient $\nabla P_0 = -\rho g \mathbf{e}_z$ (26)
and the dynamic contribution

$$\nabla P_1 = \nabla\,.\,\boldsymbol{\tau} - \rho[\boldsymbol{\omega}\wedge\mathbf{u}+\nabla(\tfrac{1}{2}u^2)]+\mathbf{j}\wedge\mathbf{B} \quad (27)$$

from the equation of motion; here $\mathbf{e}_z$ is a unit vector in the upward vertical direction, along the z axis of Cartesian co-ordinates, and $\boldsymbol{\omega}$ is the vorticity. We now investigate whether the right-hand side of (22) can be simplified by replacing P by P_0.

At this stage we set $\mathbf{B} = 0$ and consider viscous stresses, so that

$$\tau_{ij} = \rho\nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\tfrac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right), \quad (28)$$

where ν is the kinematic viscosity. If the Prandtl number is high enough for viscous forces to dominate inertial forces in (27) then

$$\frac{\langle T\mathbf{u}\,.\,\nabla P_1\rangle}{\langle T\mathbf{u}\,.\,\nabla P_0\rangle} \approx \frac{\nu T v^2/d^2}{g v \Delta T} = \frac{\nu T v}{g\Delta T d^2}, \quad (29)$$

where v is a typical velocity and the angular brackets denote horizontal averages. In I we found that for Rayleigh–Bénard convection in a Boussinesq fluid with a Rayleigh number

$$R = C_p \rho g \alpha \Delta T\, d^3/k\nu \tag{30}$$

the velocity $v \approx R^{\frac{2}{3}} k/C_p \rho d$. Assuming that this holds when $d = O(H_T)$ and substituting into (29), we find that

$$\langle T\mathbf{u}\,.\,\nabla P_1\rangle / \langle T\mathbf{u}\,.\,\nabla P_0\rangle \approx \alpha T/R^{\frac{1}{3}}. \tag{31}$$

Hence for fluids with a sufficiently small coefficient of expansion, and a sufficiently high Prandtl number for the Reynolds number to be small even at high Rayleigh numbers, ∇P_1 can be ignored in comparison with the hydrostatic pressure gradient and (22) simplifies to

$$\Phi = -\int_V \alpha T\mathbf{u}\,.\,\nabla P_0\, dV, \tag{32}$$

whence, from (26),

$$\Phi = \int_V \rho g \alpha T w\, dV, \tag{33}$$

where $w = \mathbf{u}\,.\,\mathbf{e}_z$. Moreover, if g, α and C_p are all constant (33) can be written as

$$\Phi = \frac{g\alpha}{C_p}\int dz\, A\langle \rho C_p w T\rangle, \tag{34}$$

where the bracketed term is the horizontally averaged convective heat flux and A is the area of the layer. In a vigorously convecting Boussinesq fluid, with a high Rayleigh number, nearly all the heat flux is carried by convection, except in narrow thermal boundary layers. Assuming that this remains true in deep layers we can set

$$\langle \rho C_p w T\rangle \approx f + Hz, \tag{35}$$

where f is the mean flux per unit area across the base of the layer ($z = 0$) and H is assumed constant. Then (34) becomes

$$\Phi = (g\alpha d/C_p)\,(f + \tfrac{1}{2}Hd). \tag{36}$$

If the mean heat flux per unit area across the top of the layer is $F = \mathscr{F}_u/A$ and

$$\mu = Hd/(f+Hd) = Hd/F \tag{37}$$

is the ratio of internally generated heat to the total heat flux then, from (36) and (23), the ratio

$$E = \Phi/\mathscr{F}_u \doteqdot (d/H_T)\,(1 - \tfrac{1}{2}\mu). \tag{38}$$

E is not a proper thermodynamic efficiency. Nevertheless, following Malkus (1973), we may regard E as the efficiency of conversion of heat into mechanical work within the convecting layer.

Equation (38) implies that the importance of shear-stress heating is independent of the Rayleigh number. Moreover, although we introduced viscous stresses to justify neglecting ∇P_1, (38) does not depend on the relation between τ_{ij} and u_j. In the Boussinesq limit, when $d \ll H_T$, $E \ll 1$ and shear-stress heating can be neglected in the heat flow equation; (38) is consistent with Malkus' demonstration that the shear-stress heating term is $O(d/H_T)$. Indeed, (34) can be derived for a Boussinesq fluid by following the method used by Malkus (1973) in discussing a more complicated problem. When $d = O(H_T)$, the Boussinesq approximation is no longer valid and shear-stress heating becomes important, together with variations in the adiabatic gradient.

That (38) is compatible with (8) is easily demonstrated by substituting (26) into (20) and integrating to obtain the adiabatic temperature

$$\Theta = T_u \exp[(d-z)/H_T]. \tag{39}$$

If the convection produces only small deviations from Θ then

$$\frac{\Delta T}{T_u} \gtrsim \left[\exp\left(\frac{d}{H_T}\right) - 1\right] = \frac{d}{H_T} + \frac{1}{2}\left(\frac{d}{H_T}\right)^2 + \dots. \tag{40}$$

Substituting (40) into (18) verifies that (38) is a more stringent condition, which approaches (8) in the Boussinesq limit.

The case of high Prandtl number p discussed above is appropriate for convection in the earth's mantle. It is also worth considering the form taken by (22) when viscous stresses are dominated by Reynolds stresses and $|\nabla P_1| \approx v^2/d$, so that instead of (29)

$$\langle T\mathbf{u}.\nabla P_1\rangle / \langle T\mathbf{u}.\nabla P_0\rangle \approx v^2 T/g\Delta T d. \tag{41}$$

There is as yet no reliable theory that predicts the velocity for high Reynolds number convection. We therefore consider two alternative possibilities. Suppose, first, that all the gravitational potential energy of a rising fluid element is converted into kinetic energy. Then

$$v^2 \approx g\alpha\Delta T d \tag{42}$$

and

$$\langle T\mathbf{u}.\nabla P_1\rangle / \langle T\mathbf{u}.\nabla P_0\rangle \approx \alpha T. \tag{43}$$

If, on the other hand, we suppose that the heat flux is independent of ν and d for sufficiently small values of p (Spiegel 1971 *a*, *b*) then

$$v \approx (pR)^{\frac{2}{3}} k/C_p \rho d \tag{44}$$

and

$$\langle T\mathbf{u}.\nabla P_1\rangle / \langle T\mathbf{u}.\nabla P_0\rangle \approx \alpha T (pR)^{\frac{1}{3}}. \tag{45}$$

If either (43) or (45) holds it follows that ∇P_1 can be neglected only if $\alpha T \ll 1$. Most liquids with small Prandtl numbers have $\alpha \lesssim 2 \times 10^{-4}\,{}^{\circ}\mathrm{K}^{-1}$, so this condition is generally satisfied in laboratory experiments. The shear-stress heating integral Φ is then given by (33) and the asymptotic efficiency by (38) if the relevant conditions are satisfied.

4. Numerical experiments

Although the global constraints obtained from conservation laws must always be satisfied they give no indication of the local variation of shear-stress heating within a convecting fluid, or of its possible influence on the temperature field. The numerical experiments on convection in the earth's mantle described in I showed narrow plumes at high Rayleigh numbers, suggesting that shear-stress heating could be locally important. It is therefore necessary to investigate the effects of viscous heating. Some limited understanding of its importance may be obtained from analytic solutions (Gebhart 1962; Gebhart & Mollendorf 1969; McKenzie 1969; Ackroyd 1974). A proper treatment requires a fully consistent non-Boussinesq calculation, which we shall not attempt here.

When $d/H_T \ll 1$ the leading approximation to Φ can be consistently evaluated from a solution to the Boussinesq equations. Taking numerical values from table 1 of I we find that $d/H_T \approx 0{\cdot}117$ for convection in the upper mantle; so it is reasonable to regard the shear-stress heating as a perturbation to the Boussinesq approximation with $\nabla . \mathbf{u} = 0$. For an incompressible fluid with Newtonian viscosity $\eta = \rho\nu$,

$$\tau_{ij}\frac{\partial u_i}{\partial x_j} = \tfrac{1}{2}\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2. \tag{46}$$

For two-dimensional flows there exists a stream function ψ such that

$$\mathbf{u} = (-\partial\psi/\partial z,\ 0,\ \partial\psi/\partial x) \tag{47}$$

and (48) becomes

$$\tau_{ij}\frac{\partial u_i}{\partial x_j} = \eta\left[4\left(\frac{\partial^2\psi}{\partial x\,\partial z}\right)^2+\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial z^2}\right)^2\right]. \tag{48}$$

This estimate remains valid provided that $d \ll H_T$, so that $\mathbf{u}$ is approximately solenoidal. We have computed the local shear-stress heating, given by (48), for the various models described in I, using a straightforward difference representation of the differential operators on a staggered mesh.

Contours of the viscous heating rate, together with streamlines for the appropriate flows, are shown in figure 1 for three different models with the parameter μ [defined in (37)] set equal to 0, 0·5 and 1 and a range of values for the heat flux F. These results can be used to study the local variation of the shear-stress heating rate. The tangential stress vanishes at the surface of the box within which the flow is confined. The largest values of the heating rate occur near the corners and are caused by the change in flow direction. In these regions the streamlines resemble those in figure 2 (*a*), calculated from the analytic solution for Stokes flow driven by boundaries moving parallel to themselves (Batchelor 1967). For perpendicular boundaries moving with a speed v the shear-stress heating rate

$$\tau_{ij}\frac{\partial u_i}{\partial x_j} = \frac{4v^2\eta}{r^2(1+\frac{1}{2}\pi)^2}(1+\sin 2\theta), \tag{49}$$

where (r, θ) are polar co-ordinates centred on the corner with θ measured downwards from the horizontal (McKenzie 1969; note that the term $\cos\theta$ is incorrectly given as $\cos\theta_a$ in his equation (3.17) though the correct expression was used to obtain his figure 7). Contours of the shear-stress heating rate from (49) are shown in figure 2 (*b*). They are similar to those near the corners in figure 1, though the singularity at $r = 0$ (corresponding to a discontinuous velocity on the boundary) is absent in the numerical solutions.

The most striking feature of the flows in figure 1 is the development of narrow plumes as the heat flux is increased, particularly when the layer is heated from within. As these plumes become more dominant the shear-stress heating rate

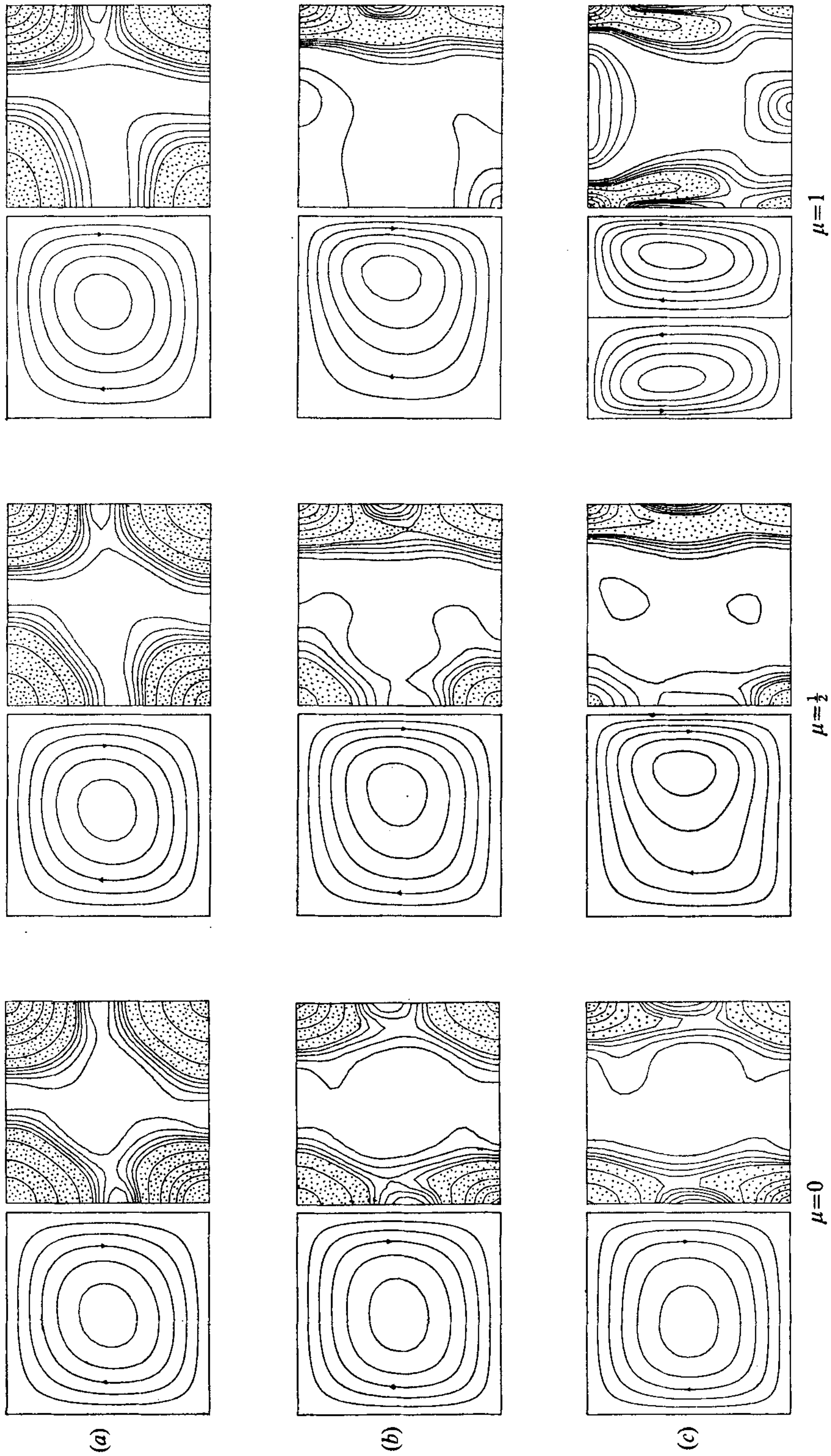
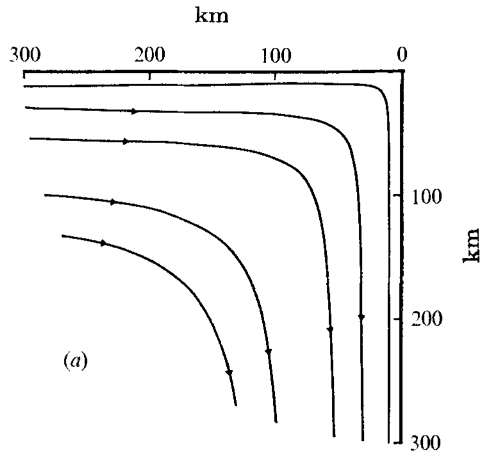


FIGURE 1. For legend see facing page.



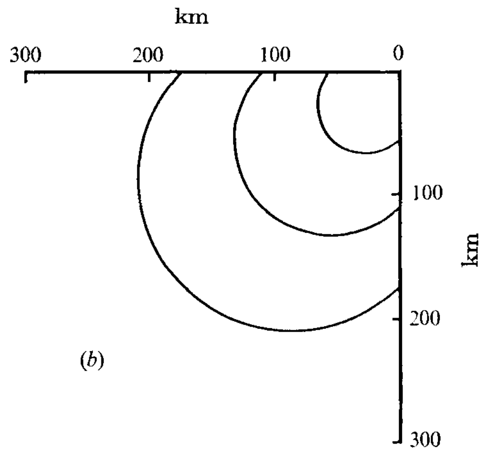


FIGURE 2. Contours of (a) the stream function and (b) the shear-stress heating rate obtained from (49). The boundaries move parallel to themselves to the right and downwards with a constant speed $v = 20$ mm and $\eta = 6 \times 10^{20}$ kg m^{-1} s^{-1}. Contour levels in (b) are $4{\cdot}8 \times 10^{-9}$, $1{\cdot}2 \times 10^{-8}$ and $4{\cdot}8 \times 10^{-8}$ Wm^{-3}, with a singularity at the corner.

FIGURE 1. Variation of the local shear-stress heating rate. Contours of the heating rate with the corresponding streamlines for convection in a Boussinesq fluid heated entirely from below ($\mu = 0$), half from within and half from below ($\mu = \frac{1}{2}$), and entirely from within ($\mu = 1$). Mean heat fluxes through the upper boundary: (a) 10^{-3} Wm^{-2}, (b) 10^{-2} Wm^{-2}, (c) $5{\cdot}85 \times 10^{-2}$ Wm^{-2}. Other parameters as in table 1 of I. Models correspond to figures 7, 9, 18, 19 and 20 of I. The contour intervals for the local shear-stress heating rate ϕ (Wm^{-3}) are not uniform. In each case contours are drawn at levels $[0{\cdot}02\ (0{\cdot}02)\ 0{\cdot}10,\ 0{\cdot}20\ (0{\cdot}20)\ 1{\cdot}00] \times \phi_m/1{\cdot}1$, where ϕ_m is the maximum heating rate, tabulated below; $\phi > \frac{1}{11}\phi_m$ in the shaded regions. Contours of the stream function ψ are equally spaced at levels $\pm[0{\cdot}0\ (0{\cdot}2)\ 1{\cdot}0] \times \psi_m/1{\cdot}1$, where $|\psi| \leqslant \psi_m$.

		$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$
$\phi_m \times 10^8$	(a)	0·077	0·068	0·058
	(b)	1·70	1·42	1·36
	(c)	10·8	12·7	7·46

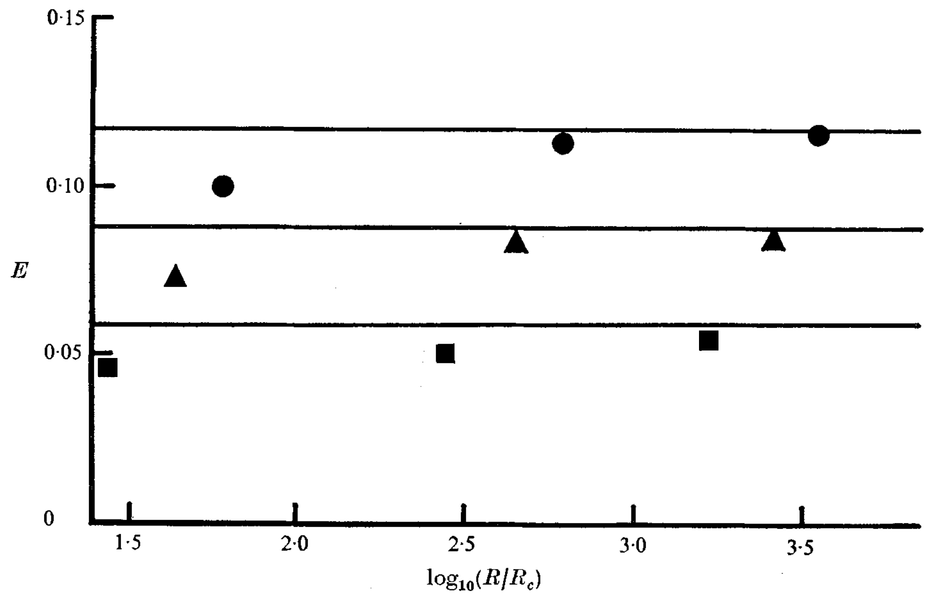


FIGURE 3. Global rate of shear-stress heating in convection: the efficiency defined by (38) as a function of Rayleigh number R. The horizontal lines are the values of E from (38) for $\mu = 0$ (top), $\mu = \frac{1}{2}$ (middle) and $\mu = 1$ (bottom). Computed values of E: ●, $\mu = 0$; ▲, $\mu = \frac{1}{2}$; ■, $\mu = 1$. Values of R corresponding to figures 1 (*a*)–(*c*) are taken from table 3 of I.

grows stronger in a narrow jet; the maximum rate increases in the corners and a local minimum appears half-way down the cell. This is particularly noticeable in the model heated entirely from within, in figure 1 (*b*); when H is further increased two cells are formed and the viscous heating becomes more diffuse. In the model heated from below T is fixed at $z = d$ and $\partial T/\partial z$ at $z = 0$ but the hot and cold plumes are nearly identical. The maximum local shear-stress heating rate occurs in figure 1 (*c*), when half the heat is generated within the box.

These results may also be used to compute the global dissipation rate Φ for comparison with the theoretical prediction of the ratio

$$E = 0{\cdot}117(1-\tfrac{1}{2}\mu) \tag{50}$$

from (38). Figure 3 shows the agreement between the numerical results and the theoretical expression, which is independent of the Rayleigh number. For the models in figure 1 (*a*) the thickness of the conducting boundary layers is not yet small compared with the depth of the layer; hence the convected flux is less than F and E is slightly less than the value in (50). Apart from these small differences E varies little while R increases by a factor of 50.

Since the shear-stress heating is concentrated in the cold descending plumes which dominate the motion it might be supposed that viscous heating could alter the convection patterns in figure 1. To check this possibility we have carried out numerical experiments with a Boussinesq fluid containing an additional variable source of heat, set equal to the local viscous dissipation rate. No perceptible changes were found in the streamlines or isotherms. We consider that the two-dimensional calculations in I would be virtually unaffected by including shear-stress heating and other non-Boussinesq effects with $d/H_T \approx 0{\cdot}1$. More generally, the gross properties of a convecting layer (Nusselt number, $\langle w \rangle$, $\langle T \rangle$) should similarly be unaffected, though specific details of a realistic problem (such as the planform or time-dependent behaviour) would be sensitive to small changes in the model.

5. Shear-stress heating in a gas

Gases differ from liquids in having a coefficient of thermal expansion that is both large and temperature dependent. For a perfect gas $\alpha = T^{-1}$, and the same expression approximates very closely to the behaviour of real gases under most laboratory conditions. Since $\alpha T = 1$ the perturbed pressure gradient ∇P_1 cannot be neglected; (22) gives

$$\Phi = -\int \mathbf{u}\,.\,\nabla P dV = g\int \rho w dV - \int \mathbf{u}\,.\,\nabla P_1 dV. \tag{51}$$

As there is no net mass flux across any surface the first integral on the right side of (51) vanishes and so

$$\Phi = -\int \mathbf{u}\,.\,\nabla P_1 dV; \tag{52}$$

i.e. the rate of shear-stress heating equals the rate of work done by the perturbed pressure forces. Thus the global dissipation rate for a gas, unlike that for a liquid, cannot be calculated from an expression similar to (38).

It is still possible to estimate Φ from (52). For high Reynolds number convection the convective velocity may be estimated from (42) and $|\nabla P_1| \approx \rho v^2/d$ from (27). Hence

$$E \approx \rho v^3/C_p \rho v \Delta T \approx d/H_T \tag{53}$$

once again. In the Boussinesq approximation the shear-stress heating rate remains negligible (Spiegel & Veronis 1960) but it becomes significant when $d \approx H_T$. This is still consistent with the results obtained in §2.

How far shear-stress heating affects convection in planetary and stellar atmospheres remains unclear. Models of stellar convective zones are generally computed using mixing-length theory (Spiegel 1971*b*, 1972) and it is generally assumed that the mixing length is comparable with the pressure scale height H_p. In that case (53) still holds, with d replaced by H_p. For a polytrope of index m the ratio $H_p/H_T = 1/(m+1)$ so $E \approx \frac{2}{5}$ for $m = \frac{3}{2}$. Thus shear-stress heating is significant but may not dominate the local structure of convection. If, as in the sun, convection extends over many scale heights the total amount of shear-stress heating can be much greater than the convective flux despite the fact that $E < 1$ for each individual eddy.

The role of shear-stress heating in a gas must be determined by numerical experiments. Since compressible convection depends critically on the form of the equation of state it is not possible to extrapolate from results for a liquid. An estimate of the effects of viscous heating can be obtained from Graham's (1975) compressible two-dimensional calculations or from computations using the anelastic approximation.

6. Ohmic heating

Consider a convectively driven hydromagnetic dynamo occupying a spherical region of radius a. We suppose that shear-stress heating is dominated by ohmic effects. (In the earth's core the magnetic Prandtl number $\mu_0 \sigma \nu \approx 0{\cdot}01$ (Gubbins 1974).) Then for a liquid, under suitable conditions, we have from (15) and (32)

$$\Phi = \int_V \frac{j^2}{\sigma} dV = -\int_V \alpha T \mathbf{u} . \nabla P_0 dV < \frac{\Delta T}{T_0} F_0 A, \tag{54}$$

where F_0 is the average flux across the surface of radius a, which is held at a fixed temperature T_0, and $A = 4\pi a^2$. From (54)

$$\Phi = \int_V \rho g \alpha T w \, dV = \int_0^a 4\pi a^2 \rho g \alpha \langle wT \rangle dr = \int_0^a \frac{g\alpha}{C_p} \mathscr{F}(r) \, dr, \tag{55}$$

where w is the radial velocity, $\mathscr{F}(r)$ is the convective flux at a radius r and angular brackets now denote averages over spherical surfaces. This corresponds to the result obtained by Malkus (1973) for a Boussinesq fluid.

To proceed further we assume that the Boussinesq approximation holds in V, so that $g(r) = (r/a) g_0$, where g_0 is the surface value of the gravitational acceleration, and that heat is generated uniformly within the sphere and removed by vigorous convection, so that $\mathscr{F}(r) = (r/a)^3 F_0 A$. Then (55) becomes

$$\Phi = \left(\frac{g_0 \alpha}{C_p}\right) F_0 A \int_0^a \left(\frac{r}{a}\right)^4 dr = \frac{1}{5}\left(\frac{g_0 \alpha a}{C_p}\right) F_0 A. \tag{56}$$

Now the adiabatic temperature difference across the sphere is given by

$$\Delta T_{\text{ad}} = \int_0^a \frac{g\alpha T}{C_p} dr \approx \frac{1}{2}\left(\frac{g_0 \alpha a}{C_p}\right) T_0 = \frac{a}{H_T} T_0 \tag{57}$$

from (20), since T does not vary significantly. Hence

$$E = \frac{\Phi}{F_0 A} = \frac{2}{5} \frac{\Delta T_{\text{ad}}}{T_0} = \frac{2}{5} \frac{a}{H_T}. \tag{58}$$

Alternatively, if all the heat is generated by a point source at the centre (as in a Cowling-model star) $\mathscr{F}(r) = F_0 A$ and (58) is replaced by

$$E = \frac{\Phi}{F_0 A} = \frac{\Delta T_{\text{ad}}}{T_0} = \frac{a}{H_T}. \tag{59}$$

Equations (58) and (59) are consistent with the expression obtained by Malkus (1973).

Hence we see that, at least in the Boussinesq limit, the efficiency is given by an expression similar to (38). If $a \gtrsim H_T$ we need to know the dependence of g on r to obtain an exact expression for Φ. However, it is clear that, so long as (55) is valid, the ratio $E = O(a/H_T)$. For a spherical region whose radius is greater than the temperature scale height the global ohmic heating rate may exceed the heat flux emerging from its surface. Since the ohmic heat is fed back into the internal energy of the liquid, without escaping from the system, the dynamo is not a heat engine doing work and there is no violation of the second law of thermodynamics. In the limit $\Delta T_{ad} \approx \Delta T \ll T_0$ the efficiency E approaches the thermodynamic value for a reversible heat engine operating with a temperature difference ΔT.

7. Geophysical implications

The discussion of ohmic heating can be applied to the earth's core. The question of whether the core is superadiabatically stratified has become controversial (Higgins & Kennedy 1971; Kennedy & Higgins 1972; Jacobs 1973; Malkus 1973). If convection occurs, the superadiabatic gradient will be small, so that $\Delta T \approx \Delta T_{ad}$; the best available estimates give a range from 3750 to 4250 °K across the outer core (Gubbins 1974). The Boussinesq approximation is valid and $\Delta T/T \approx \frac{1}{8}$. Hence the ratio of ohmic dissipation to convective flux $E \approx \frac{1}{20}$ if the convective flux increases as r^3 (corresponding to uniform heat generation). If the convective flux is uniform (as it might be if heat is liberated by accretion onto the inner core) $E \approx \frac{1}{8}$.

Conduction down the adiabatic gradient in the core contributes about 3 % of the total heat flux through the surface of the earth. If we assume that the convected heat flux is less than 10% of the flux at the earth's surface, then the total ohmic dissipation is less than about 1 %, corresponding to a rate of 3×10^{11} W (Malkus 1973). To estimate the corresponding field $\bar{B}$ in the core we suppose that the rate of ohmic heating is given by $\bar{B}^2 a/\mu_0^2 \sigma$; taking $\sigma = 5 \times 10^5\,\mathrm{mho\,m^{-1}}$ (Gubbins 1974), we find that $\bar{B} \approx 2000$ G. This upper limit is comfortably in excess of any fields that have been suggested in the core.

In previous discussions of ohmic heating in the core, e.g. by Bullard (1949), Bullard & Gellman (1954), Hide (1956) and Braginskiĭ (1964), it has been supposed that the global dissipation rate cannot exceed the total heat flux emerging from the core and that the efficiency E is limited by the efficiency of a reversible heat engine. We have shown that the ohmic heating rate may be greater than the emergent flux if $a > H_T$ and emphasized that a dynamo is not a heat engine in the thermodynamic sense. Nevertheless, since the Boussinesq approximation is valid for the core, $E \approx \Delta T/T$ and earlier results, though based on faulty arguments, are quantitatively correct.

In applying the results of §§ 3 and 4 to the earth's mantle we assume that convection is confined to the upper mantle (see the discussion in I and McKenzie & Weiss 1975) and also that a temperature scale height of 6000 km, obtained from values of α, C_p and g given in I, is correct. The principal uncertainty is in the value of α, owing to phase changes in the mantle. If the mantle material is everywhere an equilibrium assemblage then the equation of state has the form $\rho = \rho(P, T)$ regardless of whether or not phase changes are taking place. Taking account of the olivine–spinel phase change we then obtain a value $\alpha \approx 4 \times 10^{-4}$ $°\mathrm{K}^{-1}$ in the upper mantle, whence it follows that $H_T \approx d$. Consequently, realistic calculations should include all the non-Boussinesq effects. However, it is as yet uncertain whether the sinking slabs beneath island arcs do indeed contain an equilibrium assemblage, as Griggs (1972) has suggested, or a metastable one. If metastable phases are present the phase changes may have little influence on the energetics, and the Boussinesq approximation may be sufficient.

Much of the interest in shear-stress heating in the mantle arose because it offered a possible explanation for the high heat flow observed behind island arcs, particularly those of the Western Pacific (Vacquier *et al.* 1966). Though it now seems likely that the basins in which the observations were made have been produced by inter-arc spreading, as Karig (1971) suggested, the lithosphere in most of these regions is probably too old to be able to maintain the observed heat flux by continuing to cool. The problem of the origin of this heat still remains; various attempts (Oxburgh & Turcotte 1968; McKenzie 1969; Sugimura & Uyeda 1973) have been made to explain it by shear-stress heating, though none of these calculations has been concerned with the full convection problem. Although the results in figure 1 should not be used without taking the adiabatic gradient into account, they do suggest that there may be local maxima in the surface heat flux above the sinking plumes.

Another feature of interest in the models in figure 1 is the region where the rising sheet spreads out near the upper surface. The magnitude of the shear-stress heating scales as the square of the velocity as in (49) and the thermal structure will therefore be sensitive to changes in the spreading rate. Recent work on the detailed chemistry of oceanic basalts has suggested that their composition (which depends on temperature) is related to the spreading rate of the ridge that erupted them (Nisbet & Pearce 1973). These variations might be produced by local heating caused by shear stresses. Otherwise it is difficult to explain them, for the thermal structure beneath a ridge is determined by the local temperature of the mantle beneath its axis, while the spreading rate depends on forces acting over the entire surfaces of both plates.

These geophysical problems indicate the need for a consistent non-Boussinesq calculation. The anelastic approximation for a liquid is simplified, compared with that for a gas, if $\alpha T \ll 1$. A study of convection in a layer with $d \gtrsim H_T$ requires numerical experiments in which the hydrostatic variation of density, the adiabatic temperature gradient and the shear-stress heating are retained. Such a computation could confirm the results obtained here.

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